VIRTUAL AMALGAMATION OF RELATIVELY QUASICONVEX SUBGROUPS

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Abstract. For relatively hyperbolic groups, we investigate conditions guaranteeing that the subgroup generated by two relatively quasiconvex subgroups Q_1 and Q_2 is relatively quasiconvex and isomorphic to $Q_1 *_{Q_1 \cap Q_2} Q_2$. The main theorem extends results for quasiconvex subgroups of word-hyperbolic groups, and results for discrete subgroups of isometries of hyperbolic spaces.

1. Introduction

This paper continues the work that started in [6] motivated by the following question:

Problem 1. Suppose G is a relatively hyperbolic group, Q_1 and Q_2 are relatively quasiconvex subgroups of G. Investigate conditions guaranteeing that the natural homomorphism

$$Q_1 *_{Q_1 \cap Q_2} Q_2 \longrightarrow G$$

is injective and that its image $\langle Q_1 \cup Q_2 \rangle$ is relatively quasiconvex.

Let G be a group hyperbolic relative to a finite collection of subgroups $\mathbb{P}$, and let dist be a proper left invariant metric on G.

Definition 1. Two subgroups Q and R of G have *compatible parabolic subgroups* if for any maximal parabolic subgroup P of G either $Q \cap P < R \cap P$ or $R \cap P < Q \cap P$.

Theorem 2. *For any pair of relatively quasiconvex subgroups Q and R of G, there is a constant $M = M(Q, R, \mathsf{dist}) \geq 0$ with the following property. Suppose that $Q' < Q$ and $R' < R$ are subgroups such that*

(1) *$Q' \cap R'$ has finite index in $Q \cap R$,*
(2) *Q' and R' have compatible parabolic subgroups, and*
(3) *$\mathsf{dist}(1, g) \geq M$ for any g in $Q' \setminus Q' \cap R'$ or $R' \setminus Q' \cap R'$.*

Then the subgroup $\langle Q' \cup R' \rangle$ of G satisfies:

Date: March 1, 2013.

Key words and phrases. Relatively hyperbolic groups, quasiconvex subgroups, combination theorem, amalgamation.

(1) *The natural homomorphism*

$$Q' *_{Q' \cap R'} R' \longrightarrow \langle Q' \cup R' \rangle$$

is an isomorphism.

(2) *If Q' and R' are relatively quasiconvex, then so is $\langle Q' \cup R' \rangle$.*

Theorem 2 extends results by Gitik [4, Theorem 1] for word-hyperbolic groups and by the the first author [6, Theorem 1.1] for relatively hyperbolic groups, as well as the case of [1, Theorem 5.3] when Γ is geometrically finite.

Definition 3. Two subgroups Q and R of a group G can be *virtually amalgamated* if there are finite index subgroups $Q' < Q$ and $R' < R$ such that the natural map $Q' *_{Q' \cap R'} R' \longrightarrow G$ is injective.

Let Q and R be relatively quasiconvex subgroups of G, and let M be the constant provided by Theorem 2. If either G is residually finite or $Q \cap R$ is a separable subgroup of G, then there is a finite index subgroup G' of G such that $\mathsf{dist}(1, g) > M$ for every $g \in G$ with $g \notin Q \cap R$. In the case that there is such subgroup G', then the subgroups $Q' = G' \cap Q$ and $R' = G' \cap R$ are relatively quasiconvex and satisfy the hypothesis of Theorem 2; hence they have a quasiconvex virtual amalgam.

Corollary 4 (First Virtual Quasiconvex Amalgam Theorem)**.** *Let Q and R quasiconvex subgroups of G with compatible parabolic subgroups, and suppose that $Q \cap R$ is separable. Then Q and R can be virtually amalgamated in G.*

Corollary 5 (Second Virtual Quasiconvex Amalgam Theorem)**.** *Suppose that G is residually finite. Then any pair of relatively quasiconvex subgroups with compatible parabolic subgroups has a quasiconvex virtual amalgamam.*

An immediate corollary of the Virtual Quasiconvex Amalgam Theorem for residually finite relatively hyperbolic groups provides the following result by Baker-Cooper [1, Theorem 5.3].

Corollary 6 (GF subgroups have virtual amalgams.)**.** *Suppose that G is a geometrically finite subgroup of $\mathsf{Isom}(\mathbb{H}^n)$. If Q and R are subgroups of G with compatible parabolic subgroups, then Q and R have a virtual amalgam. The resulting subgroup is geometrically finite if Q and R are geometrically finite.*

2. Preliminaries

2.1. **Gromov-hyperbolic Spaces.**

Let (X, dist) be a proper and geodesic δ-hyperbolic space. Recall that a (λ, μ)–quasi-geodesic is a curve

$\gamma\colon [a,b] \to X$ parametrize by arc-length such that

$$|x-y|/\lambda - \mu \leq \mathsf{dist}(\gamma(x), \gamma(y)) \leq \lambda|x-y| + \mu$$

for all $x, y \in [a,b]$. The curve γ is a $k-$local $(\lambda,\mu)-$quasi-geodesic if the above condition is required only for $x, y \in [a,b]$ such that $|x-y| \leq k$.

Lemma 7. [3, Chapter 3, Theorem 1.2]*(Morse Lemma) For each λ, μ, δ there exists $k > 0$ with the following property. In an $\delta-$hyperbolic geodesic space, any $(\lambda,\mu)-$quasi-geodesic at k-Hausdorff-distance from the geodesic between its endpoints.*

Lemma 8. [3, Chapter 3, Theorem 1.4] *For each λ, μ, δ there exist k, λ', μ' so that any $k-$local $(\lambda,\mu)-$quasi-geodesic in a $\delta-$hyperbolic geodesic space is a $(\lambda',\mu')-$quasi-geodesic.*

Fix a basepoint $x_0 \in X$. If G is a subgroup of $Isom(X)$, we identify each element g of G with the point gx_0 of X. For $g_1, g_2 \in G$ denote by $\mathsf{dist}(g_1, g_2)$ the distance $\mathsf{dist}(g_1x_0, g_2x_0)$. Observe that if G is a discrete subgroup, this is a proper and left invariant pseudo-metric on G.

Lemma 9 (Bounded Intersection)**.** [6, Lemma 4.2] *Let G be a discrete subgroup of $\mathsf{Isom}(X)$, let Q and R be subgroups of G, and let $\mu > 0$ be a real number. Then there is a constant $M = M(Q,R,\mu) \geq 0$ so that*

$$Q \cap \mathcal{N}_\mu(R) \subset \mathcal{N}_M(Q \cap R).$$

2.2. **Relatively Quasiconvex Subgroups.** We follow the approach to relatively hyperbolic groups as developed by Hruska [5].

Definition 10 (Relative Hyperbolicity)**.** A group G is *relatively hyperbolic with respect to a finite collection of subgroups* $\mathbb{P}$ if G acts properly discontinuously and by isometries on a proper and geodesic δ-hyperbolic space X with the following property: X has a G-equivariant collection of pairwise disjoint horoballs whose union is an open set U, G acts cocompactly on $X - U$, and $\mathbb{P}$ is a set of representatives of the conjugacy classes of parabolic subgroups of G.

Throughout the rest of the paper, G is a relatively hyperbolic group acting on a proper and geodesic δ-hyperbolic space X with a G-equivariant collection of horoballs satisfying all conditions of Definition 10. As before, we fix a basepoint $x_0 \in X - U$, identify each element g of G with $gx_0 \in X$ and let $\mathsf{dist}(g_1, g_2)$ denote $\mathsf{dist}(g_1x_0, g_2x_0)$ for $g_1, g_2 \in G$.

Lemma 11. [2, Lemma 6.4]*(Cocompact actions of parabolic subgroups on thick horospheres) Let B be a horoball of X with G-stabilizer P. For any $M > 0$, P acts cocompactly on $\mathcal{N}_M(B) \cap (X - U)$.*

Lemma 12 (Parabolic Approximation)**.** *Let Q be a subgroup of G and let $\mu > 0$ be a real number. There is a constant $M = M(Q, \mu)$ with the following property. If P is a maximal parabolic subgroup of G stabilizing a horoball B, and $\{1, q\} \subset Q \cap \mathcal{N}_\mu(B)$ then there is $p \in Q \cap P$ such that $\mathsf{dist}(p, q) < M$.*

Proof. By Lemma 11, $\mathsf{dist}(q, P) < M_1$ for some constant $M_1 = M_1(Q, P)$. Then Lemma 9 implies that $\mathsf{dist}(q, Q \cap P) < M_2$ where $M_2 = N(Q, P, M_1)$. Since B is a horoball at distance less than μ from 1, there are only finitely many possibilities for B and hence for the subgroup P. Let M the maximum of all $N(Q, P, \mu)$ among the possible P. □

Definition 13 (Relatively Quasiconvex Subgroup)**.** A subgroup Q of G is *relatively quasiconvex* if there is $\mu \geq 0$ such that for any geodesic c in X with endpoints in Q, $c \cap (X - U) \subset N_\mu(Q)$.

The choice of horoballs turns out not to make a difference:

Proposition 14. [5] *If Q is relatively quasiconvex in G then for any $L \geq 0$ there is $\mu \geq 0$ such that for any geodesic c in X with endpoints in Q, $c \cap \mathcal{N}_L(X - U) \subset N_\mu(Q)$.*

3. Proof of the main theorem

For the convenience of the reader, we report below the statement of Theorem 2.

Theorem 15. *For any pair of relatively quasiconvex subgroups Q and R of G, there is a constant $M = M(Q, R, \mathsf{dist}) \geq 0$ with the following property. Suppose that $Q' < Q$ and $R' < R$ are subgroups such that*

(1) *$Q' \cap R'$ has finite index in $Q \cap R$,*
(2) *Q' and R' have compatible parabolic subgroups, and*
(3) *$\mathsf{dist}(1, g) \geq M$ for any g in $Q' \setminus Q' \cap R'$ or $R' \setminus Q' \cap R'$.*

Then the subgroup $\langle Q' \cup R' \rangle$ of G satisfies:

(1) *The natural homomorphism*
$$Q' *_{Q' \cap R'} R' \longrightarrow \langle Q' \cup R' \rangle$$
is an isomorphism.
(2) *If Q' and R' are relatively quasiconvex, then so is $\langle Q' \cup R' \rangle$.*

Consider $1 \neq g \in Q' *_{Q' \cap R'} R'$ and write it as $g = g_1 \dots g_n$ where the g_i's are alternatively elements of $Q' \backslash Q' \cap R'$ and $R' \backslash Q' \cap R'$. Moreover, assume that this product is *minimal* in the sense that $\sum \mathsf{dist}(1, g_i)$ is minimal among all such products describing g.

Claim. *(Lemma 18 below). There is a constant $K = K(Q, R, \delta)$ with the following property. For each i, let $h_i = g_1 \dots g_i$. Then the concatenation $\alpha = \alpha_1 \cdots \alpha_{n-1}$ of geodesics α_i from h_i to h_{i+1} is an $M'-$local $(1, K)-$quasi-geodesic for $M' = \min\{\mathsf{dist}(1, g_i)\}$.*

Conclusion of the proof using the claim. If we require M as in the statement of the theorem to be large enough, then we can assume $M' > k, \lambda'\mu'$ where k, λ' and μ' are as in Lemma 8 for $\lambda = 1, \mu = K$. It follows that α is a quasi-geodesic with distinct endpoints, and hence $g \neq 1$ in G. Therefore we have shown that the map $Q' *_{Q' \cap R'} R' \to G$ is injective.

It is left to prove that if Q' and R' are relatively quasiconvex, then $\langle Q', R' \rangle$ is relatively quasiconvex. By Lemma 7 (Morse Lemma), any $(\lambda', \mu')-$quasi-geodesic is at Hausdorff distance at most L from any geodesic between its endpoints. In particular, if γ is a geodesic from 1 to g, then $\gamma \cap (X - U) \subseteq \mathcal{N}_L(\alpha) \cap (X - U)$. It is enough to show that $\alpha \cap \mathcal{N}_L(X - U)$ is contained in $\mathcal{N}_\mu(\langle Q' \cup R' \rangle)$. Let $p \in \alpha \cap \mathcal{N}_L(X - U)$ and let i be so that $p \in [h_i, h_{i+1}] \cap (X - U)$. Assume $g_{i+1} \in Q'$, the other case being symmetric. As Q' is relatively quasiconvex and in view of Proposition 14, there is a constant μ so that $p \in \mathcal{N}_\mu(h_i Q') \subseteq \mathcal{N}_\mu(\langle Q' \cup R' \rangle)$ (as $h_i \in \langle Q' \cup R' \rangle$).

Proof of the Claim. The proof is a sequence of three lemmas.

Lemma 16. *Suppose $a \in Q' \cap R'$, p is a point at distance at most δ from the geodesic segment $[1, g_i g_{i+1}]$ and $\mathsf{dist}(p, g_i a) \leq M$. Then*

$$\mathsf{dist}(1, g_i) + \mathsf{dist}(1, g_{i+1}) \leq \mathsf{dist}(1, g_i g_{i+1}) + 2M + 2\delta.$$

Proof. Let $p' \in [1, g_i g_{i+1}]$ be such that $\mathsf{dist}(p, p') < \delta$. Then

$$\begin{aligned} \mathsf{dist}(1, g_i a) + \mathsf{dist}(1, a^{-1} g_{i+1}) \leq& \\ &\leq \mathsf{dist}(1, p') + \mathsf{dist}(p', g_i a) + \mathsf{dist}(g_i a, p') + \mathsf{dist}(p', g_i g_{i+1}) \\ &\leq \mathsf{dist}(1, g_i g_{i+1}) + 2M + 2\delta \end{aligned}$$

As g can be written as $g_1 \dots (g_i a)(a^{-1} g_{i+1}) \dots g_n$, the minimality assumption implies $\mathsf{dist}(1, g_i) + \mathsf{dist}(1, g_{i+1}) \leq \mathsf{dist}(1, g_i g_{i+1}) + 2M + 2\delta$. $\square$

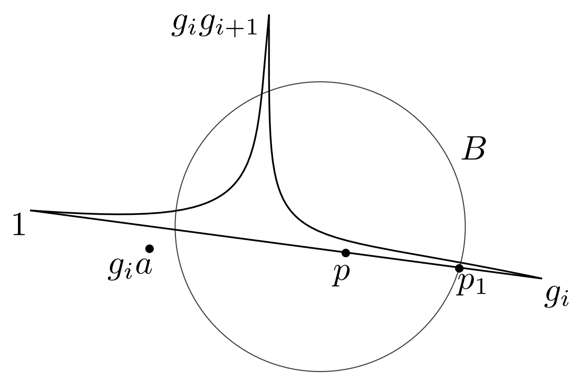
Lemma 17. *(Gromov's Inner Product is Bounded) There exists a constant $K = K(Q, R)$, not depending on g, such that*

$$\mathsf{dist}(1, g_i) + \mathsf{dist}(1, g_{i+1}) \leq \mathsf{dist}(1, g_i g_{i+1}) + K.$$

Proof. Constants which depend only on Q, R and δ are denoted by M_i, the index counts positive increments of the constant during the proof. The constant K of the statement corresponds to M_{11}.

Suppose $g_i \in Q'$, the other case being symmetric, and consider a triangle Δ with vertices $1, g_i, g_i g_{i+1}$ and let $p \in [1, g_i]$ be a center of Δ, i.e., the δ-neighborhood of p intersects all sides of Δ.

Suppose that $p \in X - U$. Then $\mathsf{dist}(p, Q), \mathsf{dist}(p, g_i R) \leq M_1$ by relative quasiconvexity of Q and R. By Lemma 9, there exists $a \in Q \cap R$ so that $\mathsf{dist}(p, g_i a) \leq M_2$. Since $Q' \cap R'$ is a finite index subgroup of $Q \cap R$, there is $b \in Q' \cap R'$ such that $\mathsf{dist}(p, g_i b) \leq M_3$. By Lemma 16, $\mathsf{dist}(1, g_i) + \mathsf{dist}(1, g_{i+1}) \leq \mathsf{dist}(1, g_i g_{i+1}) + 2M_3 + 2\delta$.



Suppose instead that p is in a horoball B, whose stabilizer is P. We can assume $\mathsf{dist}(g_i, B) \leq M_8$. Indeed, let p_1 be the entrance point of the geodesic $[g_i, 1]$ in B; then $\mathsf{dist}(p_1, Q) < M_4$ by quasiconvexity of Q. Notice that $\mathsf{dist}(p_1, [g_i, g_i g_{i+1}])$ is at most δ since p is a center of Δ and $p_1 \in [g_i, p]$. Notice that $\mathsf{dist}(p_1, [g_i, g_i g_{i+1}])$ is at most 2δ (consider a triangle with vertices p, g_i, p' for $p' \in [g_i, g_i g_{i+1}]$ so that $d(p, p') \leq \delta$). By quasiconvexity of R, there is $p_2 \in [g_i, g_i g_{i+1}]$ such that $\mathsf{dist}(p_1, p_2), \mathsf{dist}(p_2, g_i R) < M_5$. Lemma 9 implies there is $a \in Q \cap R$ such that $\mathsf{dist}(g_i a, p_1), \mathsf{dist}(g_i a, p_2) < M_6$. Since $Q' \cap R'$ is a finite index subgroup of $Q \cap R$, there is $b \in Q' \cap R'$ such that $\mathsf{dist}(g_i b, p_1), \mathsf{dist}(g_i b, p_2) < M_7$. Since g can be written as $g_1 \ldots (g_i b)(b^{-1} g_{i+1}) \ldots g_n$; by minimality

$$\begin{aligned} \mathsf{dist}(1, p_1) + \mathsf{dist}(p_1, g_i) + \mathsf{dist}(g_i, p_2) + \mathsf{dist}(p_2, g_i g_{i+1}) &= \\ = \mathsf{dist}(1, g_i) + \mathsf{dist}(1, g_i g_{i+1}) & \\ \leq \mathsf{dist}(1, g_i b) + \mathsf{dist}(1, b^{-1} g_{i+1}) & \\ = \mathsf{dist}(1, p_1) + \mathsf{dist}(p_1, g_i b) + \mathsf{dist}(g_i b, p_2) + \mathsf{dist}(p_2, g_i g_{i+1}), & \end{aligned}$$

and therefore

$$\begin{aligned} 2\,\mathsf{dist}(g_i, B) &= 2\,\mathsf{dist}(p_1, g_i) \\ &\leq \mathsf{dist}(p_1, g_i) + \mathsf{dist}(g_i, p_2) + \mathsf{dist}(p_1, p_2) \\ &\leq \mathsf{dist}(p_1, g_i b) + \mathsf{dist}(g_i b, p_2) + \mathsf{dist}(p_1, p_2) \\ &\leq 2M_8. \end{aligned}$$

Since Q' and R' have compatible parabolic subgroups, assume that $Q' \cap g_i^{-1}Pg_i \leq R' \cap g_i^{-1}Pg_i$; the other case being symmetric. By quasiconvexity of Q, there is $q_1 \in Q$ at distance M_9 from the entrance point of $[1, g_i]$ to B. By the parabolic approximation lemma applied to $\{1, g_i^{-1}q_1\} \subset Q' \cap \mathcal{N}_{M_9}(g_i^{-1}B)$, there is an element $a \in Q' \cap g_i^{-1}Pg_i$ such that $\mathsf{dist}(g_i a, q_1) \leq M_{10}$. Observe that $a \in Q' \cap R'$. By Lemma 16,

$$\mathsf{dist}(1, g_i) + \mathsf{dist}(1, g_{i+1}) \leq \mathsf{dist}(1, g_i g_{i+1}) + M_{11}. \qquad \square$$

Lemma 18. *For each i, let $h_i = g_1 \dots g_i$. Then the concatenation $\alpha = \alpha_1 \cdots \alpha_{n-1}$ of geodesics α_i from h_i to h_{i+1} is an M'−local $(1, K)$−quasi-geodesic for $M' = \min\{\mathsf{dist}(1, g_i)\}$.*

Proof. This holds in view of Lemma 17 and the following computation for $x \in [h_{i-1}, h_i]$ and $y \in [h_i, h_{i+1}]$:

$$\begin{aligned}\mathsf{dist}(h_{i-1}, x) + \mathsf{dist}(x, y) + \mathsf{dist}(y, h_{i+1}) &\geq \mathsf{dist}(h_{i-1}, h_{i+1}) \geq \\ &\geq \mathsf{dist}(h_{i-1}, h_i) + \mathsf{dist}(h_i, h_{i+1}) - K = \\ &= \mathsf{dist}(h_{i-1}, x) + \mathsf{dist}(x, h_i) + \mathsf{dist}(h_i, y) + \mathsf{dist}(y, h_{i+1}) - K\end{aligned}$$

that yields $\mathsf{dist}(x, y) + K \geq \mathsf{dist}(x, h_i) + \mathsf{dist}(h_i, y)$. $\square$

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